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OPTIMAL EDGE-COLOURINGS FOR A CLASS OF PLANAR MULTIGRAPHS

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Let G be a multigraph containing no minor isomorphic to $K_{3,3}$ or $K_5 \setminus e$ (where $K_5 \setminus e$ denotes K_5 without one of its edges). We show that the chromatic index of G is given by $\max\{\rho, \lceil\kappa\rceil\}$, where ρ is the maximum valency of G and κ is defined as

$$\max\left\{\frac{w(E(S))}{\lfloor |S|/2\rfloor} \mid S \text{ is an odd subset of vertices of } G \text{ with } |S| \geq 3\right\}$$

($w(E(S))$ being the number of edges in the subgraph induced by S). This result partially verifies a conjecture of Seymour [J. Combin. Theory (B) 31 (1981), pp. 82-94] and is actually a generalization of a result proven by Seymour [Combinatorica 10 (1990), pp. 379-392] for series-parallel graphs. It is also equivalent to the following statement: the matching polytope of a graph containing neither $K_5\setminus e$ nor $K_{3,3}$ as a minor has the integer decomposition property.

1. Introduction

In this paper we present a method for colouring the edges of multigraphs not containing a $K_5 \setminus e$ or $K_{3,3}$ minor (where $K_5 \setminus e$ denotes K_5 without one of its edges). The colourings that we construct are optimal, i.e., they contain as few colours as possible. We consider the multigraph G as a triple (V, E, w), where V is the set of vertices of G, E its set of edges and w a

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vector whose components are the edge multiplicities of G. We assume that the components of w are strictly positive and that G is without loops, and use the term "graph" when we mean "simple graph". The term "edge" may denote either a pair $\{u,v\}$ of vertices (abbreviated as uv) or a particular link between these two vertices. An edge-colouring of G is an assignment of colours to the edges of G such that no two adjacent edges (in particular, no two edges with the same endpoints) are assigned the same colour. The chromatic index, denoted by $\chi'(G)$, is the minimum number of colours in an edge-colouring of G.

In a 1965 paper, Vizing [14] proved that the chromatic index of a graph G is either ρ or $\rho+1$ (where ρ denotes the maximum degree of G). He also proved that the chromatic index of a multigraph G is at most $\rho+m$, where m denotes the maximum edge multiplicity of G (see also Bondy and Murty [2]). This bound can be improved in certain cases, but there are few classes of graphs for which we know the exact value of $\chi'(G)$. Actually, good bounds for $\chi'(G)$ seem to depend less on edge multiplicities than on the number of edges in certain subgraphs of G. We shall now describe a "good" lower bound for $\chi'(G)$.

For a subset F of edges of the multigraph G, $w(F)$ denotes $\sum_{e\in F} w_e$, and for a vertex v of G, $\delta(v)$ denotes the star of v (i.e., the set $\{e \in E \,|\, v \text{ is an endpoint of } e\}$). For a subset S of vertices of G, $E(S)$ denotes the set of edges with both ends in S, and $\delta(S)$ the coboundary of S, i.e., the set $\{uv \in E \,|\, u \in S \text{ and } v \in V \setminus S\}$. We shall distinguish between degree and valency, that is, $|\delta(v)|$ is the degree of v while $w(\delta(v))$ is the valency of v. Obviously $\chi'(G)$ is at least equal to the maximum valency of G, denoted ρ. On the other hand, for any edge-colouring of G and any odd subset S of V with $|S| \geq 3$, at most $\lfloor |S|/2 \rfloor$ edges in $E(S)$ have the same colour (where $\lfloor r \rfloor$ denotes the greatest integer n such that $n \leq r$). Thus if $|S|$ is odd and greater than 1, $\chi'(G)$ is at least equal to $\frac{w(E(S))}{\lfloor |S|/2 \rfloor} = \frac{2w(E(S))}{|S|-1}$. Finally, if κ (the *maximum odd set quotient* of G) denotes

$$\max\left\{\frac{w(E(S))}{\lfloor |S|/2 \rfloor} \,|\, S \subseteq V, |S| \text{ odd and } |S| \geq 3\right\},$$

$\chi'(G)$ is at least $\max\{\rho, \lceil \kappa \rceil\}$ (where $\lceil r \rceil$ denotes $-\lfloor -r \rfloor$).

The example of the Petersen graph shows that the previous bound is not always tight (for this graph $\chi'(G) = 4$ but $\max\{\rho, \lceil \kappa \rceil\} = 3$). On the other hand, we do not know of any graph or multigraph for which $\chi'(G)$ is greater than $\max\{\rho+1, \lceil \kappa \rceil\}$. This has led Seymour to conjecture (in [11]) that $\chi'(G) = \max\{\rho+1, \lceil \kappa \rceil\}$ for all multigraphs (see also Goldberg [4]). Seymour also conjectured that $\chi'(G) = \max\{\rho, \lceil \kappa \rceil\}$ for planar multigraphs (see [12]).

Note that the second conjecture (henceforth called the *exact conjecture*) is a generalization of the four-colour theorem, and that Seymour [13] has verified it for series-parallel multigraphs (i.e., multigraphs with no K_4 minor). At about the same time, Marcotte [6] proved the first conjecture for multigraphs with no minor isomorphic to $K_5 \setminus e$.

The Petersen graph minus one vertex also has chromatic index 4. The truth of Tutte's edge-colouring conjecture would imply that every "snark" (i.e., every cubic 3-connected graph with chromatic index 4) contains a homeomorph of this graph, and hence a homeomorph of $K_{3,3}$ (see Isaacs [5] and Robertson *et al.* [10]). In the same vein, Plantholt and Tipnis [8] have shown that a multigraph with ten or fewer vertices satisfies the exact conjecture if the underlying simple graph does not contain the Petersen graph minus one vertex. Thus we have to exclude $K_{3,3}$ if we wish to prove that $\chi'(G)=\max\{\rho,\lceil\kappa\rceil\}$ for all multigraphs G belonging to a given class. If we exclude both $K_{3,3}$ and K_5, the statement "$\chi'(G)=\max\{\rho,\lceil\kappa\rceil\}$" is precisely the exact conjecture. In this paper, we prove that the exact conjecture holds if we exclude $K_{3,3}$ and $K_5 \setminus e$ (note that it is not enough to exclude $K_5 \setminus e$, because the Petersen graph minus one vertex has no $K_5 \setminus e$ minor).

We recall that by a theorem of Wagner [15], a connected graph with no $K_5 \setminus e$ minor and no 1-vertex or 2-vertex cutset is isomorphic to K_n (for $1 \le n \le 3$), the prism, $K_{3,3}$ or the n-wheel for some $n \ge 3$. Hence a multigraph with no $K_5 \setminus e$ or $K_{3,3}$ minor contains a 1-vertex or 2-vertex cutset, **or** its underlying graph is isomorphic to K_n (for $1 \le n \le 3$), the prism or the n-wheel for some $n \ge 3$. If the multigraph G has a 2-vertex cutset with at least two vertices on each side, we can apply the decomposition technique described in Marcotte [7] (see also Section 6 of the present paper). It remains to prove that $\chi'(G)$ is equal to $\max\{\rho,\lceil\kappa\rceil\}$ for multigraphs G whose 2-vertex cutsets (if any) separate one vertex from the rest of the multigraph. The simple graphs underlying these multigraphs, if they contain a K_4 minor, are obtained from the prism or the n-wheel by replacing each edge by a path of length 1 or 2 or a triangle. A precise definition is given below.

Definition 1.1. Let $H=(X,F)$ be a graph. We say that the graph S is a *triangle-subdivision* of H if

1. the vertex set of S is $X \cup T$, where T is a set of new vertices of degree 2, and
2. the edge set of S is $\cup_{uv\in F} P(u,v)$, where for each edge uv of H, $P(u,v)=\{uv\},\{ut,tv\}$ or $\{ut,tv,uv\}$ for some vertex t in T.

If the graph underlying $G=(V,E,w)$ is a triangle-subdivision of the triangle, the prism or the n-wheel, we say that G itself is a triangle-subdivision of the triangle, the prism or the n-wheel. Figure 1 represents a triangle-subdivision of the 10-wheel (note that an edge whose multiplicity is not shown has multiplicity 1).

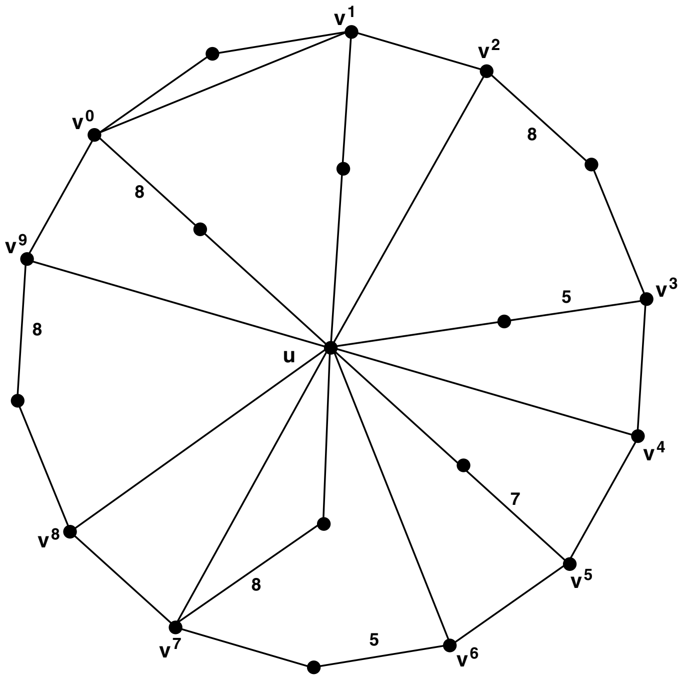

Fig. 1 A triangle-subdivision of the 10-wheel

In the next section, we describe the relationship between the edge-colouring problem and matching theory, and give definitions used throughout the paper. In Section 3, we describe some reductions that can be applied to the graphs we are trying to colour. Sections 4 and 5 describe methods for colouring triangle-subdivisions of the n-wheel and the prism, respectively. Finally, we prove in Section 6 that the chromatic index of a multigraph with no $K_5 \setminus e$ or $K_{3,3}$ minor is equal to $\max\{\rho, \lceil\kappa\rceil\}$.

2. Edge-colourings and matching theory

By definition of edge-colouring, the edges that are assigned a given colour form a matching. Thus our problem amounts to covering the edges of a multigraph with matchings, and it is natural to investigate the relationship between edge-colourings and matching theory. For instance, results of Pulleyblank and Edmonds [9] suggest that we need not consider all subsets of odd cardinality when computing κ, and we now introduce the subsets that "matter". For any subset S of V, G_S denotes the subgraph of G induced by S (we shall henceforth use the term "subgraph" instead of "submultigraph"). For any vertex v of S, a matching N in G_S *saturates* v if v belongs to an edge of N; otherwise, N leaves v *exposed* (or *unmatched*). If N saturates v, the *mate* of v is the only vertex u such that vu belongs to N. Finally, if $|S|$ is odd and greater than 1 and N leaves only v exposed, we say that N is a *near perfect matching* of G_S (or S) *deficient* at v.

Definition 2.1. Let $G=(V,E,w)$ be a multigraph, and S a subset of V with $|S|$ odd and $|S|\geq 3$. The multigraph G_S (or S itself) is *critical* if for every vertex v in S, there exists a near perfect matching of G_S deficient at v.

As noted above, if E is nonempty, the edge-colouring problem for $G=(V,E,w)$ can be formulated as a covering program, i.e., the program

$$\begin{array}{rcll} \min 1.\lambda^t & & & \\ \text{s.t. } \lambda A & = & w & \qquad (IP) \\ \lambda & \geq & 0 & \\ \lambda_i & \in & Z \text{ for each i,} & \end{array}$$

where λ is a row vector and the rows of A are the incidence vectors of matchings in G. Let (LP) denote the linear relaxation of (IP). Observe that for a given positive real number r, (LP) has a feasible solution of value r if and only if $\frac{w}{r}$ is a convex combination of incidence vectors of matchings. It follows from a theorem of Edmonds [3] that (LP) has a feasible solution of value r if and only if

$$\left(\frac{w}{r}\right)(\delta(v)) \leq 1 \text{ for every vertex } v \text{ of } G$$

$$\text{and } \left(\frac{w}{r}\right)(E(S)) \leq \lfloor |S|/2 \rfloor \text{ for every subset } S \text{ of } V \text{ with } |S| \text{ odd and } |S| \geq 3,$$

that is, if and only if r is at least $\max\{\rho,\kappa\}$.

We conclude that the optimal value of (LP) is $\max\{\rho,\kappa\}$, the fractional chromatic index of G, and the exact conjecture is equivalent to the statement that (IP) has the integer rounding property for any planar multigraph G. The latter, in turn, is equivalent to the statement that the matching polytope of a planar graph has the *integer decomposition property* (see Baum and Trotter [1]). On the other hand, the characterization of the facets of matching polytopes by Pulleyblank and Edmonds [9] implies that (LP) has a feasible solution of value r if and only if

$$\left(\frac{w}{r}\right)(\delta(v)) \leq 1 \text{ for every vertex v of } G$$

and

$$\left(\frac{w}{r}\right)(E(S)) \leq \lfloor |S|/2 \rfloor \text{ for every subset } S \text{ of } V$$

inducing a 2-connected critical subgraph of G.

Hence the fractional chromatic index of G is also equal to $\max\{\rho, \kappa'\}$, where κ' is defined as

$$\max\left\{\frac{w(E(S))}{\lfloor |S|/2 \rfloor} \mid S \text{ induces a 2-connected critical subgraph of } G\right\}.$$

Observe that in the n-wheel, the number of 2-connected induced subgraphs is a quadratic function of n, and we do not encounter great difficulties in proving the exact conjecture for wheels. In the same vein, in order to prove that $\chi'(G) = \max\{\rho+1, \lceil\kappa\rceil\}$ for the multigraph $G = (S, E, w)$, it suffices to consider the case where $\kappa > \rho$ and $\kappa = \frac{w(E)}{\lfloor |S|/2 \rfloor}$. For a triangle-subdivision of the n-wheel having this property, the number of 2-connected induced subgraphs with $\kappa > \rho$ is a quadratic function of n (in the worst case), and the proof of the conjecture is relatively easy. The problem we address in this paper, however, seems much more difficult, because we cannot restrict our attention to a "quadratic" number of subgraphs. In a triangle-subdivision of the n-wheel, the number of induced subgraphs that are critical and 2-connected grows exponentially as a function of n, and since we cannot disregard the subgraphs for which κ is at most ρ, finding an appropriate colouring is more difficult than in the previous case.

In the rest of the paper $k(G)$ will denote $\max\{\rho, \lceil\kappa\rceil\}$, and $G \setminus M$ the multigraph obtained from G by removing the matching M. In order to prove the exact conjecture (i.e., $\chi'(G) = k(G)$), we shall use induction on $k(G)$ and the number of vertices of G. In some cases (to be studied in Section 3), we can split G into two subgraphs of "small" chromatic index and combine their edge-colourings to colour the edges of G. This decomposition, called *shrinking*, is suggested by the matching algorithm and allows the use of induction on the number of vertices. When no subset of vertices can be "shrunk", we construct a matching M such that $k(G \setminus M) = k(G) - 1$ holds and we use induction on $k(G)$. In Sections 4 and 5 we describe such matchings for triangle-subdivisions of the n-wheel and the prism, respectively.

3. Reductions

In this section, we show how to decompose a multigraph G containing certain types of subgraphs. We only assume that G is connected (in particular, we do not forbid any minor). Note that in the rest of the paper, "subgraph" will always mean "vertex-induced subgraph" and "odd set" will always mean "odd set of cardinality greater than 1".

Definition 3.1. Let $G=(V,E,w)$ be a multigraph and S a subset of V such that $2\leq |S|<|V|$ and G_S is connected. By *shrinking* S we mean replacing G by the multigraphs G_S and $G_P=(X,F^1,w^1)$, where

- $X=(V\setminus S)\cup\{v_P\}$ and v_P is a new vertex,
- $F^1=E(V\setminus S)\cup\{vv_P|$ there exists $vu\in E$ such that $v\in V\setminus S$ and $u\in S\}$,
- $w^1_e=w_e$ for $e\in E(V\setminus S)$, and
- $w^1_e=w(\delta(v,S))$ if e is of the form vv_P and $\delta(v,S)$ denotes $\{vu\in E\,|\,u\in S\}$.

Note that if G does not contain a given minor, G_P cannot contain this minor either. In the following proofs we use the same notation as in Definition 3.1.

Lemma 3.2. *Let $G=(V,E,w)$ be a connected multigraph, and S an odd subset of V such that $3\leq |S|<|V|$, G_S is connected and $w(E(S))=k(G)\lfloor |S|/2\rfloor$. Let G_P denote the graph obtained from G by shrinking S. Then $k(G_P)\leq k(G)$, and $\chi'(G)=k(G)$ if $\chi'(G_P)$ and $\chi'(G_S)$ are at most $k(G)$.*

Proof. The valency of a vertex $v\neq v_P$ is the same in G_P as in G, and thus at most ρ. The valency of v_P is given by

$$\begin{aligned} w^1(\delta(v_P)) &= w(\delta(S)) \\ &= \sum_{u\in S} w(\delta(u)) - 2w(E(S)) \\ &\leq \rho|S| - 2k(G)\lfloor |S|/2\rfloor \\ &\leq \rho. \end{aligned}$$

We conclude that the maximum valency of G_P is at most ρ. Let us now consider an odd subset T of vertices of G_P. If T does not contain v_P, $w^1(E(T))$ is equal to $w(E(T))$ and thus at most $k(G)\lfloor |T|/2\rfloor$. If T contains v_P, we observe that

$$\begin{aligned} w^1(E(T)) &= w(E((T\setminus\{v_P\})\cup S)) - w(E(S)) \\ &\leq k(G)\lfloor (|T|+|S|-1)/2\rfloor - k(G)\lfloor |S|/2\rfloor \\ &= k(G)\lfloor |T|/2\rfloor. \end{aligned}$$

We have shown that $k(G_P) \le k(G)$.

On the other hand, if $\chi'(G_P)$ and $\chi'(G_S)$ are at most $k(G)$, let $C_1 = \{L_1, L_2, \ldots, L_{k(G)}\}$ and $C_2 = \{N_1, N_2, \ldots, N_{k(G)}\}$ be edge-colourings of G_P and G_S respectively (where the L_i and N_j are (possibly empty) matchings). Since $w(E(S)) = k(G)\lfloor |S|/2 \rfloor$, each N_j is a near perfect matching of G_S. Let t denote $w(\delta(S))$ (that is, $w^1(\delta(v_P))$), and v^1u^1, v^2u^2, $\ldots$, v^tu^t a list of the edges of $\delta(S)$ (where the u^ℓ belong to S). Note that for a given u in S, the number of i such that $u \in v^iu^i$ is at most $k(G) - w^2(\delta(u))$ (where w^2 denotes the restriction of w to G_S). Also the number of matchings deficient at u in C_2 is exactly $k(G) - w^2(\delta(u))$. Hence we may assume, without loss of generality, that L_i contains edge v^iv_P and N_i is a near perfect matching deficient at u^i for $i = 1, 2, \ldots, t$. If we define M_i as $(L_i \setminus \{v^iv_P\}) \cup N_i \cup \{v^iu^i\}$ for $i = 1, 2, \ldots, t$ and as $L_i \cup N_i$ for $i = t+1, \ldots, k(G)$, $\{M_1, M_2, \ldots, M_{k(G)}\}$ is clearly an edge-colouring of G. ∎

Note that in the previous lemma, the subset S that we shrink need not be 2-connected or even critical. The following corollary is very useful for "reducing" a graph G containing vertices of degree 2 (for instance, a triangle-subdivision of a 3-connected graph).

Corollary 3.3. *Let $G = (V, E, w)$ be a connected multigraph, and u a vertex of G of degree 2 and valency $k(G)$. Let G_P be the multigraph obtained from G by shrinking $\{u, v^1, v^2\}$, where v^1 and v^2 are the neighbours of u. Then $k(G_P) \le k(G)$, and $\chi'(G) = k(G)$ if $\chi'(G_P) \le k(G)$.*

Proof. Take $S = \{u, v^1, v^2\}$ in Lemma 3.2. ∎

A multigraph $G = (S, E, w)$ such that $w(E) = k(G)\lfloor |S|/2 \rfloor$ will be called a *full* multigraph. If necessary we can go one step further than Lemma 3.2 and shrink some subgraphs that are "nearly full", i.e., whose odd set quotients are slightly less than $k(G)$. For instance, in Figure 1, the subgraph induced by u, v^7 and their common neighbour verifies the hypotheses of the following lemma. Note that if the odd set quotient of G_T is less than $k(G)$ and we replace T by the pseudo-vertex v_P, the valency of v_P may be greater than $k(G)$. Thus we shrink T only if $w(\delta(T))$ is at most $k(G)$. We let $N(S)$ denote the set $\{v \in V \mid v \notin S$ and $\delta(v)$ is contained in $\delta(S)\}$.

Lemma 3.4. *Let $G = (V, E, w)$ be a connected multigraph whose maximum odd set quotient is less than $k(G)$, and assume that there exists an odd set S such that*

- *$|S| \ge 3$, $V \setminus (S \cup N(S))$ is nonempty and S and $V \setminus (S \cup N(S))$ induce connected subgraphs of G,*

- *$w(E(S)) = k(G)\lfloor |S|/2 \rfloor - 1$, and*
- *$w(\delta(S)) \le k(G)+1$ and $w(\delta(S \cup N(S))) \le k(G)$.*

We let G_P (resp. H) denote the multigraph obtained from G by shrinking $S \cup N(S)$ (resp. $V \setminus (S \cup N(S))$). Then $k(G_P)$ and $k(H)$ are at most $k(G)$, and $\chi'(G) = k(G)$ if $\chi'(G_P)$ and $\chi'(H)$ are at most $k(G)$.

Proof. We first consider $G_P = (X, F^1, w^1)$, and note that the valency of a vertex $v \neq v_P$ is the same in G_P as in G, and thus at most ρ. On the other hand, $w^1(\delta(v_P)) = w(\delta(S \cup N(S))) \le k(G)$ by assumption. As in Lemma 3.2, we have $w^1(E(T)) = w(E(T)) \le k(G)\lfloor |T|/2 \rfloor$ for an odd set T not containing v_P. Finally, if T does contain v_P, we have

$$\begin{aligned} w^1(E(T)) &= w(E((T \setminus \{v_P\}) \cup S)) - w(E(S)) \\ &\le k(G)\lfloor (|T| + |S| - 1)/2 \rfloor - 1 - \{k(G)\lfloor |S|/2 \rfloor - 1\} \\ &= k(G)\lfloor |T|/2 \rfloor. \end{aligned}$$

The inequality obtains because the maximum odd set quotient of G is less than $k(G)$.

We now consider the graph H. Let w^2 denote the vector of edge multiplicities of H, and v^0 the vertex that "replaces" $V \setminus (S \cup N(S))$ in H. We first show that the valency of every vertex of H is at most $k(G)$. This is obvious for any vertex v different from v^0, since the valency of v is the same in H as in G. As for v^0, $w^2(\delta(v^0)) = w(\delta(S \cup N(S))) \le k(G)$ by assumption.

Let T be an odd subset of vertices of H such that $|T| \ge 3$ and $v^0 \notin T$. Then $w^2(E(T))$ is equal to $w(E(T))$ and thus at most $k(G)\lfloor |T|/2 \rfloor$. If T contains v^0, we use induction on $|T \cap N(S)|$ to prove that $w^2(E(T))$ is at most $k(G)\lfloor |T|/2 \rfloor$ (note that $N(S)$ does not include v^0).

- If $T \cap N(S) = \emptyset$, we have

  $$w^2(E(S)) + w^2(\delta(S)) = \sum_{u \in S} w^2(\delta(u)) - w^2(E(S))$$

  and

  $$w^2(E(S \setminus T)) + w^2(\delta(S \setminus T)) = \sum_{u \in S \setminus T} w^2(\delta(u)) - w^2(E(S \setminus T)),$$

  which yields

  $$\begin{aligned} w^2(E(T)) &\le \{w^2(E(S)) + w^2(\delta(S))\} - \{w^2(E(S \setminus T)) + w^2(\delta(S \setminus T))\} \\ &= \sum_{u \in S} w^2(\delta(u)) - w^2(E(S)) - \sum_{u \in S \setminus T} w^2(\delta(u)) + w^2(E(S \setminus T)) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{u\in T\setminus\{v^0\}} w^2(\delta(u)) - w^2(E(S)) + w^2(E(S\setminus T)) \\
&\le k(G)(|T|-1) - \{k(G)\lfloor |S|/2\rfloor - 1\} + k(G)\lfloor |S\setminus T|/2\rfloor - 1 \\
&= k(G)\lfloor |T|/2\rfloor.
\end{aligned}
$$

The second inequality obtains because the maximum odd set quotient of G is less than $k(G)$.

- If $T\cap N(S)=\{v^1\}$ for some v^1, we have

$$
\begin{aligned}
w^2(E(T)) &\le w^2(\delta(v^0)) + w^2(\delta(v^1)) + w(E(T\setminus\{v^0,v^1\})) \\
&\le k(G)+1+k(G)\lfloor |T\setminus\{v^0,v^1\}|/2\rfloor - 1 \\
&= k(G)\lfloor |T|/2\rfloor.
\end{aligned}
$$

The second inequality obtains because $w(\delta(S))\le k(G)+1$ and the maximum odd set quotient of G is less than $k(G)$.

- If $|T\cap N(S)|\ge 2$, we let v^1 and v^2 denote two vertices in $T\cap N(S)$ and obtain

$$
\begin{aligned}
w^2(E(T)) &\le w^2(\delta(v^1)) + w^2(\delta(v^2)) + w(E(T\setminus\{v^1,v^2\})) \\
&\le w^2(\delta(S)) - w^2(\delta(v^0)) + k(G)\lfloor |T\setminus\{v^1,v^2\}|/2\rfloor \\
&\qquad \text{since } w(E(T\setminus\{v^1,v^2\})) \le k(G)\lfloor |T\setminus\{v^1,v^2\}|/2\rfloor \\
&\qquad \text{by the induction hypothesis} \\
&\le k(G)+1-w^2(\delta(v^0)) + k(G)\lfloor |T\setminus\{v^1,v^2\}|/2\rfloor \\
&\le k(G)\lfloor |T|/2\rfloor \qquad \text{because } G \text{ is connected.}
\end{aligned}
$$

We have thus proved that $k(G_P)$ and $k(H)$ are at most $k(G)$. In order to show that $\chi'(G)=k(G)$ if $\chi'(G_P)$ and $\chi'(H)$ are at most $k(G)$, we observe that in any colourings of G_P and H, the links in $\delta(v_P)$ and $\delta(v^0)$ must be assigned different colours. Since $w^1(\delta(v_P))=w^2(\delta(v^0))$, we can identify the colour of each link in $\delta(v_P)$ with the colour of the corresponding link in $\delta(v^0)$ to obtain a colouring of G (this argument can also be found in Theorem 1.2 of Marcotte [6]). ∎

Corollary 3.5. *Let $G=(V,E,w)$ be a connected multigraph whose maximum odd set quotient is less than $k(G)$, and assume that there exists an odd set S such that*

- *$|S|\ge 3$, $V\setminus(S\cup N(S))$ is nonempty and S and $V\setminus(S\cup N(S))$ induce connected subgraphs of G,*
- *$w(E(S))=k(G)\lfloor |S|/2\rfloor - 1$, and*
- *S contains two vertices of valency at most $k(G)-1$, or S contains one such vertex and $N(S)$ contains at least one vertex.*

Let G_P and H be defined as in Lemma 3.4. Then $k(G_P)$ and $k(H)$ are at most $k(G)$, and $\chi'(G)=k(G)$ if $\chi'(G_P)$ and $\chi'(H)$ are at most $k(G)$.

Proof. The corollary will follow from Lemma 3.4 if we can show that the hypothesis "$w(\delta(S))\leq k(G)+1$ and $w(\delta(S\cup N(S)))\leq k(G)$" holds. Suppose first that S contains a vertex of valency at most $k(G)-1$ (denoted by v'), and $N(S)$ contains a vertex (denoted by v^*). Then

$$\begin{aligned} w(\delta(S)) &= \sum_{v\in S} w(\delta(v)) - 2w(E(S)) \\ &= \sum_{v\in S\setminus\{v'\}} w(\delta(v)) + w(\delta(v')) - 2w(E(S)) \\ &\leq k(G)(|S|-1) + k(G) - 1 - 2\{k(G)\lfloor |S|/2\rfloor - 1\} \\ &= k(G)|S| - 1 - \{k(G)(|S|-1) - 2\} \\ &= k(G)+1. \end{aligned}$$

Further, since G is connected, v^* is of valency at least 1 and we have

$$w(\delta(S\cup N(S))) \leq w(\delta(S)) - w(\delta(v^*)) \leq k(G).$$

The conclusion of Lemma 3.4 (and Corollary 3.5) follows.

If S contains two vertices of valency at most $k(G)-1$, we conclude by a similar argument that $w(\delta(S))$, and hence $w(\delta(S\cup N(S)))$, is at most $k(G)$. ∎

If G satisfies the hypotheses of Lemma 3.2 or 3.4 and the exact conjecture holds for graphs with fewer vertices than G, we can conclude that $\chi'(G)=k(G)$. Hence in Sections 4 and 5 we consider graphs that do not satisfy the hypotheses of these lemmas.

Definition 3.6. Let us assume that the multigraph $G=(V,E,w)$ has an odd subset S of vertices with the following properties:

- $|S|\geq 3$ and S induces a 2-connected critical subgraph of G,
- $V=S\cup N(S)$, and
- $w(E(S))=k(G)\lfloor |S|/2\rfloor -1$.

Then we say that G is *nearly full*.

Definition 3.7. Let $G=(V,E,w)$ be a multigraph and $\mathcal{C}$ the collection $\{S\,|\,S$ is an odd subset of V, $|S|\geq 3$ and G_S is connected$\}$. G is *reduced* if it has the following properties:

- if $w(E(S))=k(G)\lfloor |S|/2\rfloor$ for some S in $\mathcal{C}$, then S is equal to V (i.e., G is a full multigraph), and

- if the maximum odd set quotient of G is strictly less than $k(G)$, and for some S in $\mathcal{C}$, $w(E(S))$ is equal to $k(G)\lfloor |S|/2\rfloor - 1$, $w(\delta(S))$ is at most $k(G)+1$, $w(\delta(S\cup N(S)))$ is at most $k(G)$ and $V\setminus(S\cup N(S))$ induces a connected subgraph of G, **then** V is equal to $S\cup N(S)$.

Note that by Corollary 3.3, if G is reduced, every vertex of degree 2 in G is of valency at most $k(G)-1$.

4. Wheels

Let $G=(V,E,w)$ be a reduced multigraph that is a triangle-subdivision of the n-wheel for some $n\geq 3$. The vertices of degree at least 3 (the *main vertices*) will be denoted by $u, v^0, v^1, \ldots, v^{n-1}$, where u is the *center* of the wheel. The vertex of degree 2 that is adjacent to u and v^i (resp. v^i and v^{i+1}) will be denoted by $t(u,v^i)$ (resp. $t(v^i,v^{i+1})$). Of course, not all vertices of degree 2 need belong to G. Note that in an expression like v^{i+1}, the "+" sign represents addition modulo n. A vertex of the form $t(u,v^i)$ (resp. $t(v^i,v^{i+1})$) is said to be *inner* (resp. *outer*). Finally, for $i\neq j$, we let $B(v^i,v^j)$ denote the set $\{v^i,v^{i+1},\ldots,v^j\}$ and $BT(v^i,v^j)$ the set $B(v^i,v^j)\cup\{t(v^k,v^{k+1}) \mid k=i,\ldots,j-1\}\cap V$. Clearly, if S induces a 2-connected subgraph of G that is not a triangle, the intersection of S and $\{u,v^0,v^1,\ldots,v^{n-1}\}$ is either $\{v^0,v^1,\ldots,v^{n-1}\}$ or a set of the form $\{u\}\cup B(v^i,v^j)$.

Our goal is to find a matching (or "colour") to be removed from the reduced multigraph G. Note that G does not always have a perfect or near perfect matching, and that even if it does, such a matching is not necessarily the "correct" one. If G is full, the "correct" matching leaves only one vertex exposed, but in general, it will be necessary to discard some vertices of G in order to construct this matching. We now define the notions of "sector" and "standard matching". Note that for a subgraph H of G containing all the v^i, $OC(H)$ (the *outer cycle* of H) denotes the set of vertices of H of the form v^i or $t(v^i,v^{i+1})$.

Definition 4.1. Let $G=(V,E,w)$ be a triangle-subdivision of the n-wheel for some $n\geq 3$, and H a subgraph of G containing all the main vertices. A *sector* is a subset of vertices of H of the form $\{u,t(u,v^k),t(u,v^\ell)\}\cup(BT(v^k,v^\ell)\cap OC(H))$, where $k\neq\ell$ and H does not contain any of the vertices $t(u,v^i)$ for $v^i\in B(v^k,v^\ell)\setminus\{v^k,v^\ell\}$. A sector is said to be *odd* if it contains an odd number of vertices, and *even* otherwise.

For instance, the set $\{u,t(u,v^1),t(u,v^3),v^1,v^2,t(v^2,v^3),v^3\}$ is a sector of the graph depicted in Figure 1. In the above definition H need not be a triangle-subdivision of a wheel (for instance H is a K_4-free graph if u is of degree 2 in H). Note also that if H has an even sector, it does not contain a near perfect matching deficient at u. Thus a multigraph H is critical only if it has no even sector. Finally, we observe that the definition of sector can be extended to 2-connected subgraphs of G whose intersection with the set of main vertices is of the form $\{u\} \cup B(v^i, v^j)$.

Definition 4.2. Let $G=(V,E,w)$ be a triangle-subdivision of the n-wheel for some $n \geq 3$, and H a subgraph of G containing all the main vertices. Assume that $OC(H)$ induces a 2-connected subgraph and H has at least one sector (that is, at least two inner vertices). A *standard H-matching M* in G is constructed as follows:

- *Case 1*: H has at least one even sector:
  Let k be the smallest index such that $B(v^k, v^\ell)$ determines an even sector for some ℓ. M contains the edge $\{t(u,v^\ell), u\}$ and the (unique) perfect matching of $BT(v^k, v^\ell) \setminus \{v^k\}$. For all the other sectors the edges of M are selected according to the following rule: M contains the (unique) perfect matching of $BT(v^i, v^j) \setminus \{v^i\}$ if the sector determined by $B(v^i, v^j)$ is even, and the (unique) perfect matching of $(BT(v^i, v^j) \setminus \{v^i\}) \cup \{t(u, v^j)\}$ if this sector is odd.
- *Case 2*: H has no even sector:
  Let F be the subgraph obtained from H by removing a vertex of the form $t(u,v^i)$ (say, $t(u,v^k)$). If F contains a sector, it contains an even one and we define M to be a standard F-matching (in the sense of Case 1). If F contains a single vertex of the form $t(u,v^i)$ (say, $t(u,v^\ell)$), we choose M to be any perfect matching of F containing edge $\{u, t(u,v^\ell)\}$. In both cases M is a near perfect matching of H deficient at $t(u,v^k)$.

In Figure 2 H is the subgraph obtained by discarding vertices $t(v^0, v^1)$ and $t(u, v^1)$. The set $B(v^0, v^3)$ determines an even sector in H and the boldface edges represent a standard H-matching in G. Note that in a standard H-matching, all the main vertices are saturated and u is matched to an inner vertex. Also, if H is critical, any standard H-matching is a near perfect matching of H. To prove the exact conjecture for triangle-subdivisions of the n-wheel, we consider first the case of full graphs.

Theorem 4.3. *Let $G=(S,E,w)$ be a triangle-subdivision of the n-wheel for some $n \geq 3$, and assume that G is critical, reduced and full. There exists a matching M in G such that the fractional chromatic index of $G \setminus M$ (and thus $k(G \setminus M)$) is at most $k(G)-1$.*

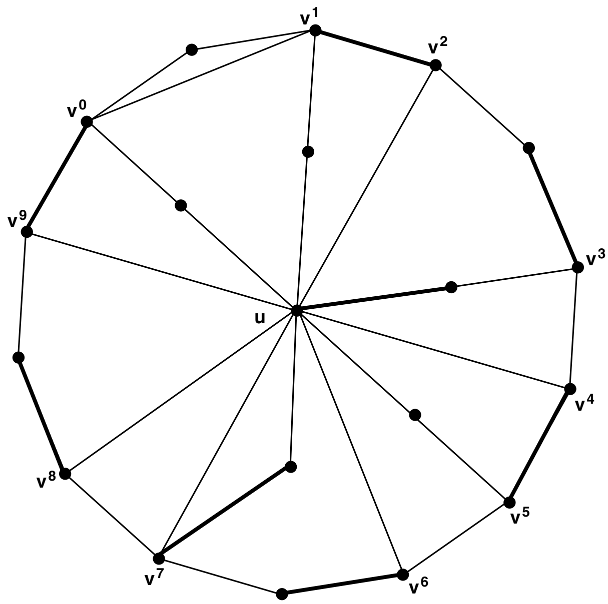

Fig. 2 A standard H-matching in the graph represented in Figure 1

Proof. Since the only multigraph with $k(G)=3$ is the simple 3-wheel (that is, K_4), we may assume that $k(G)$ is at least 4. If G has at least one sector, we choose M to be any standard G-matching in G. If G has no sector but at least one vertex of degree 2 (say, v), we choose M to be any near perfect matching of G deficient at v. If G has no vertex of degree 2, we note that the valency of some vertex (say, v^*) must be strictly less than $k(G)$ (otherwise $2w(E(S))=\sum_{v\in S} w(\delta(v))$ would be equal to $k(G)|S|$, and thus greater than $2k(G)\lfloor |S|/2\rfloor$). We then take M to be a near perfect matching of G deficient at v^*.

Let y be the incidence vector of M. By construction, if G contains a vertex of degree 2, M saturates all the main vertices. Since G is reduced, every vertex of degree two has valency less than $k(G)$. Therefore $(w-y)(\delta(v)) \leq k(G)-1$ for every v if G contains at least one vertex of degree 2. If not, M is a near perfect matching deficient at a vertex of valency less than $k(G)$. We conclude that in this case also, the maximum valency of $G\setminus M$ is at most $k(G)-1$.

We now show that $(w-y)(E(T)) \leq (k(G)-1)\lfloor |T|/2\rfloor$ for every odd subset T of S. By the remarks of Section 2, it suffices to verify this inequality for sets T inducing 2-connected critical subgraphs of G. The proof is by induction on $|T|$. Note that since G is reduced, $w(E(T)) < k(G)\lfloor |T|/2 \rfloor$ for any odd subset T of S with $|T| < |S|$. Thus $(w-y)(E(T)) \leq (k(G)-1)\lfloor |T|/2 \rfloor$ if $|T|=3$. Also $y(E(S)) = y(E) = \lfloor |S|/2 \rfloor$ since M is a near perfect matching of G. Hence $(w-y)(E(S)) = (k(G)-1)\lfloor |S|/2 \rfloor$, and from now on we assume that $5 \leq |T| < |S|$.

In the remainder of the proof, the gist of the argument is that $M \cap E(T)$ leaves few vertices of T exposed (this will imply that $(w-y)(E(T))$ is not "too large"). Because M is a near perfect matching, proving that $M \cap E(T)$ leaves few vertices exposed amounts to proving that $M \cap \delta(T)$ contains few edges. We can classify the sets T according to the "shape" of $M \cap \delta(T)$. If S happens to be $T \cup N(T)$, $M \cap \delta(T)$ will consist of edges of $\delta(N(T))$ only. Otherwise, if T does not contain u, $M \cap \delta(T)$ will consist of edges of $\delta(N(T))$ and edges containing u or an inner vertex. Finally, if T contains u, the set of main vertices of T must be of the form $\{u\} \cup B(v^k, v^\ell)$ (since T induces a 2-connected subgraph), and $M \cap \delta(T)$ will consist of edges of $\delta(N(T))$ and edges containing u, v^k or v^ℓ.

These observations lead one to think that if u does not belong to T, $M \cap \delta(T)$ could contain far too many edges (one per inner vertex!). Fortunately, we can show that when there are many inner vertices, the "weight" of T cannot be large.

Claim 4.4. *If u does not belong to T and G has at least 5 inner vertices, $(w-y)(E(T))$ is at most $(k(G)-1)\lfloor |T|/2 \rfloor$.*

Proof. Observe that for any two vertices of degree 2, say, v and v', we have $w(\delta(v)) + w(\delta(v')) \geq k(G)$ (otherwise the odd set quotient of the subgraph induced by $S \backslash \{v, v'\}$ would be greater than $k(G)$). Let U be a set of precisely 5 inner vertices including the mate of u (note that this mate is an inner vertex since G has more than one sector). Since $w(\delta(v)) + w(\delta(v')) \geq k(G)$ for any vertices v and v' of degree 2, $\sum_{v \in U} w(\delta(v))$ is at least $5/2k(G)$. On the other hand, at most 4 edges of M are of the form $\{v^i, t(u, v^i)\}$ with $t(u, v^i)$ belonging to U. Thus

$$\begin{aligned}
(w-y)(\delta(T)) &\geq w(\delta(T,U)) - y(\delta(T,U)) \\
&\geq \sum_{v \in U} w(\delta(v)) - w(\delta(u)) - y(\delta(T,U)) \\
&\geq 5/2k(G) - k(G) - 4 \\
&\geq k(G) - 2.
\end{aligned}$$

The third inequality obtains because $w(\delta(u)) \leq k(G)$ and the fourth because $k(G) \geq 4$.

Now

$$2(w-y)(E(T)) = \sum_{v\in T}(w-y)(\delta(v)) - (w-y)(\delta(T))$$
$$\leq (k(G)-1)|T| - k(G) + 2$$
$$= (k(G)-1)(|T|-1) + 1,$$

from which we conclude that $(w-y)(E(T)) \leq (k(G)-1)\lfloor |T|/2 \rfloor$. ∎

Claim 4.4 allows us to assume that there are at most 4 inner vertices in the neighbourhood of a set T not containing u. Since u is matched to an inner vertex whenever G has a sector, $\delta(T\cup N(T))$ is a set of at most three edges (all of them of the form $\{v^i, t(u,v^i)\}$ if G has at least one sector). Therefore $|M\cap\delta(T)| = |M\cap\delta(N(T))| + |M\cap\delta(T\cup N(T))|$ is at most $|N(T)|+3$. The same relation holds when u belongs to T, because any edge of $\delta(T\cup N(T))$ contains u, v^k or v^ℓ (where the intersection of T and $\{u, v^0, v^1, \ldots, v^{n-1}\}$ is $\{u\}\cup B(v^k, v^\ell)$). In the remainder of the proof, we consider only sets T such that T induces a 2-connected critical subgraph of G and $|M\cap\delta(T)|$ is at most $|N(T)|+3$.

We now prove by induction on $|T|$ that $(w-y)(E(T)) \leq (k(G)-1)\lfloor |T|/2 \rfloor$. The only exposed vertex of G is denoted v^*.

Case A. $|N(T)| \leq 1$ and $v^* \notin T$.

Then $|M\cap\delta(T)|$ is at most 4 (since $|M\cap\delta(T)| \leq |N(T)|+3$), and because all the vertices in T are saturated by M, $|M\cap E(T)| = (|T| - |M\cap\delta(T)|)/2$. We conclude that $|M\cap E(T)| \geq \lfloor |T|/2 \rfloor - 1$ since $|M\cap E(T)|$ is an integer. Hence

$$(w-y)(E(T)) = w(E(T)) - |M\cap E(T)|$$
$$\leq k(G)\lfloor |T|/2 \rfloor - 1 - |M\cap E(T)|$$
$$\leq (k(G)-1)\lfloor |T|/2 \rfloor.$$

The first inequality follows from the assumption that G is reduced.

Case B. $N(T) = \emptyset$ and $v^* \in T$.

Then $|M\cap\delta(T)|$ is at most 3, and from $|M\cap E(T)| = (|T| - 1 - |M\cap\delta(T)|)/2$ we conclude again that $|M\cap E(T)|$ is at least $\lfloor |T|/2 \rfloor - 1$ and $(w-y)(E(T))$ at most $(k(G)-1)\lfloor |T|/2 \rfloor$.

Case C. $|N(T)| = 1$ and $v^* \in T$.

Let W denote $S\setminus(T\cup N(T))$, F the subgraph induced by W and v' the only vertex in $N(T)$. Since T is 2-connected, the 2-connected subgraphs of F (if any) must be triangles, and by the remarks of Section 2 the fractional chromatic index of $F\backslash(M\cap E(W))$ is at most $k(G)-1$. Therefore $w'(E(W))$ is at most $(k(G)-1)\lfloor |W|/2\rfloor$ if we let w' stand for $w-y$. We have

$$\begin{aligned}
2w'(E(T)) &= \sum_{v\in T} w'(\delta(v)) - w'(\delta(W)) - w'(\delta(v'))\\
&= \sum_{v\in T} w'(\delta(v)) - \left\{\sum_{v\in W} w'(\delta(v)) - 2w'(E(W))\right\} - w'(\delta(v'))\\
&= \sum_{v\in T} w'(\delta(v)) + 2w'(E(W)) - \left\{\sum_{v\in S} w'(\delta(v)) - \sum_{v\in S\backslash(W\cup\{v'\})} w'(\delta(v))\right\}\\
&= \sum_{v\in T} w'(\delta(v)) + 2w'(E(W)) - (k(G)-1)(|S|-1) + \sum_{v\in T} w'(\delta(v))\\
&\leq 2(k(G)-1)|T| + (k(G)-1)(|W|-1) - (k(G)-1)(|S|-1)\\
&= (k(G)-1)(|T|-1).
\end{aligned}$$

Case D. $|N(T)|\geq 2$.

Let v and v' be any two vertices in $N(T)$. Then $w(\delta(T))\geq w(\delta(v))+w(\delta(v'))\geq k(G)$, and thus $(w-y)(\delta(T))\geq k(G)-2$. This implies that $(w-y)(E(T))$ is at most $(k(G)-1)\lfloor |T|/2\rfloor$ by the same argument as in Claim 4.4.

This completes the proof of the exact conjecture for full graphs that are triangle-subdivisions of the n-wheel. ∎

The proof of Theorem 4.3 relies (among other facts) on the observation that $w(\delta(v))+w(\delta(v'))\geq k(G)$ for any pair v, v' of vertices of degree 2. In general, this inequality does not hold and we must discard some vertices of "small" valency, more precisely, of valency at most $(k(G)-1)/2$. If H (the resulting multigraph) contains at least one sector, the required matching will be a standard H-matching in G.

Theorem 4.5. *Let $G=(V,E,w)$ be a triangle-subdivision of the n-wheel, and assume that G is reduced and not full. There exists a matching M in G such that the fractional chromatic index of $G\backslash M$ (and thus $k(G\backslash M)$) is at most $k(G)-1$.*

Proof. Since the only multigraph with $k(G)=3$ is the simple 3-wheel (that is, K_4), we may assume that $k(G)$ is at least 4. Let H be the graph obtained from G by removing the inner vertices of valency at most $(k(G)-1)/2$, and the outer vertices of valency at most $(k(G)-1)/2$ whose neighbours are joined by an edge. If H contains a sector, we choose M to be a standard H-matching in G. Note that if the multiplicity of edge $\{v^7,t(u,v^7)\}$ were equal to 7 in the multigraph of Figure 1, the multigraph would be reduced and the matching in Figure 2 would be an illustration of this case. If H does not contain a sector but contains a perfect matching or a near perfect matching deficient at a vertex of degree 2, we choose M to be such a matching. Note that M can be chosen in this way if H has a vertex of the form $t(v^i, v^{i+1})$ and another of the form $t(u, v^j)$, or an edge of the form uv^i and another of the form $\{u, t(u, v^j)\}$.

Finally, if H has no sector, no perfect matching and no near perfect matching deficient at a vertex of degree 2, one of the following three cases must occur.

- H contains one edge of the form $\{u, t(u, v^j)\}$ but no edge of the form uv^i Then G contains another inner vertex (say, $t(u, v^k)$), and we choose M to be a perfect matching of $H \cup \{t(u, v^k)\}$.
- H contains an edge of the form uv^i but no edge of the form $\{u, t(u, v^j)\}$ In this case H has an odd number of vertices. If G contains an inner vertex (say, $t(u, v^k)$), we choose M to be a perfect matching of $H \cup \{t(u, v^k)\}$ containing the edge $\{u, t(u, v^k)\}$ (where $H \cup \{t(u, v^k)\}$ denotes the subgraph induced by $t(u, v^k)$ and the vertices of H). If G has no inner vertex but at least one outer vertex (say, v), we choose M to be a perfect matching of $H \cup \{v\}$. Otherwise, H is identical to G and we choose M to be a near perfect matching of G deficient at a vertex of valency at most $k(G) - 1$.
- H contains no edge incident upon u Then G has two inner vertices, say, $t(u, v^i)$ and $t(u, v^j)$. We choose M to be a perfect matching of $H \cup \{t(u, v^i)\}$ if H has an odd number of vertices, and a perfect matching of $H \cup \{t(u, v^i), t(u, v^j)\}$ otherwise.

Let y be the incidence vector of M. By construction M saturates all vertices of G of valency $k(G)$, and we conclude that $(w - y)(\delta(v)) \leq k(G) - 1$ by the same argument as in Theorem 4.3. To prove that M is the required matching, we also need to verify that $(w - y)(E(S))$ is at most $(k(G) - 1)\lfloor |S|/2 \rfloor$ for any odd set S. Actually, we shall prove the following (stronger) statement: if L denotes the restriction of M to H and z the incidence vector of L, $(w - z)(E(S))$ is at most $(k(G) - 1)\lfloor |S|/2 \rfloor$ for any odd set S. Let $N_h(S)$ denote the set $\{v \text{ is a vertex of } H \,|\, v \notin S \text{ and } \delta(v) \subseteq \delta(S)\}$.

As in the proof of Theorem 4.3, we assume that $|S|$ is greater than 3 and S induces a 2-connected critical subgraph of G. The proof is by induction on $|S|$. Observe that $exp(S)$, the number of vertices of S left exposed by $L \cap E(S)$, is at most $|S \setminus V(H)| + |L \cap \delta(N(S))| + |L \cap \delta(S \cup N(S))| + |U|$, where $V(H)$ denotes the set of vertices of H and U the set of vertices in

$S \cap V(H)$ left exposed by L. Since L is included in H, $exp(S)$ is at most $|S \setminus V(H)| + |N_h(S)| + |L \cap \delta(S \cup N(S))| + |U|$. If S contains u, the set of main vertices in S is of the form $\{u\} \cup B(v^k, v^\ell)$, and $exp(S)$ is at most $|S \setminus V(H)| + |N_h(S)| + |U \setminus \{u\}| + 3$ because any edge of $\delta(S \cup N(S))$ contains u, v^k or v^ℓ. The following claim will be used in the case where u belongs to S (note that $S \cup N(S)$ might or might not be equal to V).

Claim 4.6. *Let S be a subset of vertices inducing a 2-connected critical subgraph of G, and assume that $S \cap \{u, v^0, v^1, \ldots, v^{n-1}\}$ is of the form $\{u\} \cup B(v^k, v^\ell)$. If H contains an even sector and $|S \setminus V(H)| + |N_h(S)|$ is equal to 1 (resp. 0), the cardinality of $U \setminus \{u\}$ is at most 2 (resp. 1). If H does not contain an even sector, L leaves at most two vertices of S exposed (i.e., $|U| \leq 2$), and if there are exactly two such vertices, one of them must be u.*

Proof. Let us assume that H contains an even sector, $|S \setminus V(H)| + |N_h(S)|$ is equal to 1 and $U \setminus \{u\}$ contains at least three vertices. By definition of U and construction of M, these vertices belong to $S \cap V(H)$ and to even sectors of H. They are thus of the form $t(u, v^i)$. Let $t(u, v^j)$, $t(u, v^k)$ and $t(u, v^\ell)$ be three such vertices, where $j < k < \ell$. Since G_S is 2-connected and v^j, v^k and v^ℓ belong to S, S contains all the main vertices of at least two of the even sectors to which $t(u, v^j)$, $t(u, v^k)$ and $t(u, v^\ell)$ belong. Because $|S \setminus V(H)| + |N_h(S)|$ is equal to 1, S must contain **all** the vertices of at least one even sector, contradicting the assumption that G_S is critical. Therefore $U \setminus \{u\}$ contains at most two vertices. If H contains an even sector and $|S \setminus V(H)| + |N_h(S)|$ is equal to 0, a similar argument shows that $U \setminus \{u\}$ is at most 1.

If H does not contain an even sector, we can check easily that by construction of M, at most two vertices of H are left exposed by M or are matched to vertices not belonging to H. Further, if there are two such vertices, one of them must be u. It follows that $|U|$ (the number of vertices of $S \cap V(H)$ left exposed by L) is at most 2. ∎

If u belongs to S and $S \cup N(S)$ is equal to V (in particular, if S contains all the main vertices of G), $exp(S)$ is at most $|S \setminus V(H)| + |N_h(S)| + |U|$. The following claim will be used in the case where $S \cup N(S)$ is equal to V, whether u belongs to S or not. We leave its proof to the reader because it is very similar to that of the previous claim.

Claim 4.7. *Let S be a subset of vertices inducing a 2-connected critical subgraph of G. If $S \cup N(S)$ is equal to V and $|S \setminus V(H)| + |N_h(S)|$ at most 2, L leaves at most two vertices of S exposed (i.e., $|U| \leq 2$).*

Finally, if S does not contain u, $L\cap\delta(S\cup N(S))$ could contain as many edges as there are inner vertices in H. As in Theorem 4.3, we can prove that when there are many such vertices, the "weight" of S cannot be large. We note that for any two inner vertices belonging to H, say, v and v', we have $w(\delta(v))+w(\delta(v'))\geq k(G)$ (this is a trivial consequence of the definition of H). The proof of the following claim is identical to that of Claim 4.4 and we shall omit it.

Claim 4.8. *If u does not belong to S and H has at least 5 inner vertices, $(w-y)(E(S))$ is at most $(k(G)-1)\lfloor |S|/2\rfloor$.*

Claim 4.8 allows us to assume that there are at most 4 inner vertices in the neighbourhood of a set S not containing u. If H has at least one sector, we know that M is contained in H, that u is matched to an inner vertex and that the only vertices left exposed by M are inner vertices. Since L is identical to M, we conclude that $U=\emptyset$ and $exp(S)$ is at most $|S\setminus V(H)|+|N_h(S)|+3$. If H does not have a sector but contains a perfect matching or a near perfect matching deficient at a vertex of degree 2, $|U|$ is at most 1 and $|L\cap\delta(S\cup N(S))|$ at most 2 (because H has at most one inner vertex). Finally, if H contains no sector, no perfect matching and no near perfect matching deficient at a vertex of degree 2, it can be checked easily that $|L\cap\delta(S\cup N(S))|+|U|$ is at most 2. In the rest of the proof we shall thus assume that $exp(S)$ is at most $|S\setminus V(H)|+|N_h(S)|+3$ whenever $u\notin S$.

We now show that $(w-z)(E(S))$ is at most $(k(G)-1)\lfloor |S|/2\rfloor$ for any odd set S, except in Case E and the last subcase of Case C, where we show directly that $(w-y)(E(S))$ is at most $(k(G)-1)\lfloor |S|/2\rfloor$.

Case A. $S\setminus V(H)=\emptyset$ and $N_h(S)=\emptyset$.

If $u\notin S$, $exp(S)$ is at most 3. If $u\in S$, $U\setminus\{u\}$ contains at most one vertex by Claim 4.6, and thus $exp(S)$ is at most 4. Then $|L\cap E(S)|=(|S|-exp(S))/2$ is at least $\lceil(|S|-4)/2\rceil=\lfloor |S|/2\rfloor-1$, and $(w-z)(E(S))$ is at most $(k(G)-1)\lfloor |S|/2\rfloor$ by the same argument as in Case A of Theorem 4.3.

Case B. $|S\setminus V(H)|+|N_h(S)|=1$ and $w(E(S))\leq k(G)\lfloor |S|/2\rfloor-2$.

If $u\notin S$, $exp(S)$ is at most 4. If $u\in S$, $U\setminus\{u\}$ contains at most two vertices by Claim 4.6, and thus $exp(S)$ is at most 6. We conclude that $|L\cap E(S)|$ is at least $\lfloor |S|/2\rfloor-2$ and $(w-z)(E(S))$ at most $(k(G)-1)\lfloor |S|/2\rfloor$ by an argument similar to that in Case A.

Case C. $|S\setminus V(H)|+|N_h(S)|=1$ and $w(E(S))=k(G)\lfloor |S|/2\rfloor-1$.

If $u\notin S$, we may use the same argument as in the previous two cases since $exp(S)$ is at most 4. If $u\in S$ and $S\cup N(S)$ is equal to V, $|U|$ is at most 2 by Claim 4.7. Hence $exp(S)$ is at most 3 and $(w-z)(E(S))$ at most $(k(G)-1)\lfloor |S|/2\rfloor$.

We turn to the case where $u \in S$ and $S \cup N(S)$ is not equal to V. Assume first that $|N_h(S)| = 1$. The subgraph induced by $V \setminus (S \cup N(S))$ must be connected and contain more than one vertex. Thus if S contained a vertex of valency at most $k(G)-1$, we could apply Corollary 3.5 to "decompose" G into smaller multigraphs. Since G is reduced, we may conclude that S has no vertex of valency at most $k(G)-1$ and hence no inner vertex. Because S is 2-connected and included in H, H contains two edges of the form uv^i and a vertex of degree 2 (the only vertex in $N_h(S)$). By construction M is included in H and identical to L, which allows us to conclude that $U \setminus \{u\}$ is empty, $exp(S)$ at most 4 and $(w-z)(E(S))$ at most $(k(G)-1)\lfloor |S|/2 \rfloor$.

Suppose now that $|S \setminus V(H)| = 1$. If $N(S)$ were not empty, we could again apply Corollary 3.5 to decompose G (this follows because the only vertex in $S \setminus V(H)$ is of degree 2). The hypothesis that G is reduced thus implies that $N(S)$ is empty and all vertices of $S \cap V(H)$ are of valency $k(G)$. Since M saturates all vertices of valency $k(G)$, $M \cap E(S)$ leaves at most 4 vertices of S exposed and $(w-y)(E(S))$ is at most $(k(G)-1)\lfloor |S|/2 \rfloor$.

Case D. $|S \setminus V(H)| \geq 2$.

Then S contains two vertices of valency at most $(k(G)-1)/2$, say a and b (note that all vertices removed from G have valency at most $(k(G)-1)/2$). By the induction hypothesis, $(w-z)(E(S \setminus \{a,b\})) \leq (k(G)-1)\lfloor (|S|-2)/2 \rfloor$, and thus

$$(w-z)(E(S)) \leq (k(G)-1)\lfloor (|S|-2)/2 \rfloor + 2(k(G)-1)/2 = (k(G)-1)\lfloor |S|/2 \rfloor.$$

Case E. $|S \setminus V(H)| = 1$, $|N_h(S)| \geq 1$ and $N_h(S)$ contains a vertex of valency greater than $(k(G)-1)/2$.

Let a denote the only vertex in $S \setminus V(H)$, and b a vertex in $N_h(S)$ whose valency is greater than $(k(G)-1)/2$. Then

$$\begin{aligned}
2(w-y)(E(S)) &= \sum_{v \in S}(w-y)(\delta(v)) - (w-y)(\delta(S)) \\
&\leq (k(G)-1)(|S|-1) + (w-y)(\delta(a)) - (w-y)(\delta(b)) \\
&\qquad \text{since the maximum valency of } G \setminus M \text{ is at most } k(G)-1 \\
&\leq (k(G)-1)(|S|-1) + (k(G)-1)/2 - (w-y)(\delta(b)) \\
&\qquad \text{since } a \in S \setminus V(H) \\
&< (k(G)-1)(|S|-1) + y(\delta(b)) \qquad \text{by choice of } b \\
&\leq (k(G)-1)(|S|-1) + 1 \\
&\qquad \text{since } y \text{ is the incidence vector of a matching.}
\end{aligned}$$

We conclude that $(w-y)(E(S))$ is at most $(k(G)-1)\lfloor |S|/2 \rfloor$.

Case F. $|S \setminus V(H)| = 1$, $|N_h(S)| \geq 1$ and $N_h(S)$ does not contain a vertex of valency greater than $(k(G)-1)/2$.

Observe that at most one vertex in $N_h(S)$ may be of valency at most $(k(G)-1)/2$. If there were two such vertices, they would be outer vertices or main vertices. For instance, if $t(v^i, v^{i+1})$ and $t(v^j, v^{j+1})$ belonged to $N_h(S)$ and were of valency at most $(k(G)-1)/2$, the definition of H would imply that G does not contain edges $v^i v^{i+1}$ and $v^j v^{j+1}$. But then S would not induce a 2-connected subgraph, contradicting our assumption. We also reach a contradiction if we assume that $N_h(S)$ contains two main vertices (or an outer vertex and a main vertex) of valency at most $(k(G)-1)/2$.

Therefore $|N_h(S)| = 1$. If the only vertex in $N_h(S)$ is an outer vertex, S must contain all the main vertices because it induces a 2-connected subgraph of G. Hence $S \cup N(S) = V$. If the only vertex in $N_h(S)$ is a main vertex, $S \cup N_h(S)$ contains all the main vertices and thus $S \cup N(S)$ is equal to V also. Since $|S \setminus V(H)| + |N_h(S)| = 2$, $|U|$ is at most 2 by Claim 4.7 and $exp(S) = |S \setminus V(H)| + |N_h(S)| + |U|$ is at most 4. We conclude that $(w–z)(E(S))$ is at most $(k(G)-1)\lfloor |S|/2 \rfloor$.

Case G. $S \setminus V(H) = \emptyset$ and $|N_h(S)| \geq 2$.

Let us assume first that $N_h(S)$ contains two vertices v and v' of valency greater than $(k(G)-1)/2$. Then

$$\begin{aligned}(w-z)(\delta(S)) &\geq (w-z)(\delta(v)) + (w-z)(\delta(v')) \\ &\geq (w(\delta(v)) - 1) + (w(\delta(v')) - 1) \\ &\geq k(G) - 2.\end{aligned}$$

But this implies readily that $(w-z)(E(S)) \leq (k(G)-1)\lfloor |S|/2 \rfloor$ (see Case D of Theorem 4.3).

Finally, if $N_h(S)$ contains at most one vertex of valency greater than $(k(G)-1)/2$, $|N_h(S)|$ must be equal to 2 since by the argument in Case F, $N_h(S)$ cannot contain more than one vertex of valency at most $(k(G)-1)/2$. It follows easily that $S \cup N(S)$ is equal to V and $|U|$ is at most 2 (by Claim 4.7). We conclude that $(w-z)(E(S))$ is at most $(k(G)-1)\lfloor |S|/2 \rfloor$ by the same argument as in Case F. ∎

5. The prism

Let $G = (V, E, w)$ be a triangle-subdivision of the prism and $V^* = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ the set of main vertices of G (i.e., vertices of G of degree at least 3). Without loss of generality, we assume that $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$ induce the two “triangles” of the prism, and that $v_1 v_4$, $v_2 v_5$ and

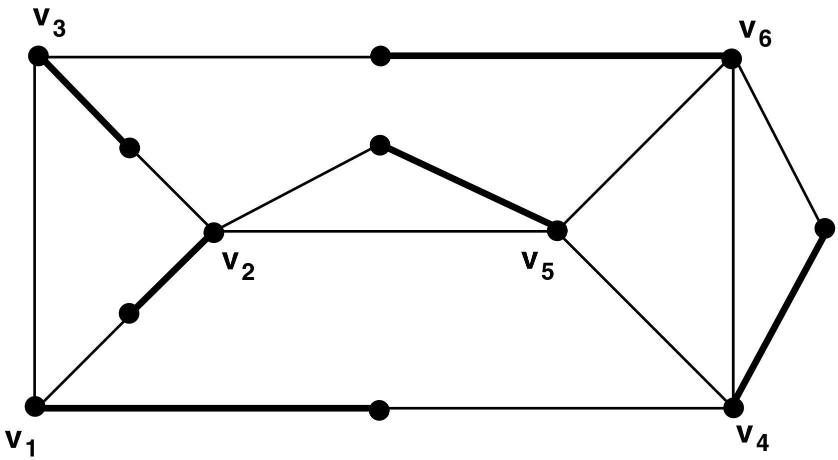

Fig. 3 A triangle-subdivision of the prism and one of its standard matchings

v_3v_6 are its three "matching edges" (see Figure 3 for an illustration, where the boldface edges form a perfect matching of G). We shall denote by v_{ij} the only vertex of degree 2 (if any) that is adjacent to v_i and v_j. The subgraph induced by $\{v_1,v_2,v_3\}\cup(V\cap\{v_{12},v_{13},v_{23}\})$ is denoted by $\langle v_1v_2v_3\rangle$, and $\langle v_4v_5v_6\rangle$ is defined similarly. The subgraph induced by $\{v_i,v_j,v_{ij}\}$ (if v_{ij} is a vertex of G) or $\{v_i,v_j\}$ (if not) will be called the *pseudo-edge* v_iv_j and denoted by $\langle v_iv_j\rangle$. Finally, we say that pseudo-edges $\langle v_iv_j\rangle$ and $\langle v_kv_\ell\rangle$ are *opposite edges* if they belong to different triangles and there is only one pseudo-edge joining a vertex of $\langle v_iv_j\rangle$ to a vertex of $\langle v_kv_\ell\rangle$.

The following lemma describes the kind of matching that will be used to find an edge-colouring of G when G is full.

Lemma 5.1. *Let $G=(V,E,w)$ be a triangle-subdivision of the prism. If G is critical, there exists a matching M in G with the following properties:*

- *M is a near perfect matching of G,*
- *M saturates all the main vertices, and*
- *$|M\cap\delta(T\cup N(T))|$ is at most 3 for any subset T of V inducing a 2-connected subgraph of G.*

Proof. We observe that neither triangle of the prism may contain three vertices of degree 2. If $\langle v_1v_2v_3\rangle$, say, contained three such vertices, G would not have a near perfect matching deficient at v_1, because the vertices of degree 2 could be matched to v_2 and v_3 only. We consider the following cases, which exhaust all possibilities since G has an odd number of vertices.

- If at least one of the two triangles contains a vertex of degree 2, G must have two opposite edges (say, $\langle v_1v_2\rangle$ and $\langle v_5v_6\rangle$) such that $\langle v_1v_2\rangle$ does not contain a vertex of degree 2 while $\langle v_5v_6\rangle$ does. Let H be the graph obtained from G by removing the vertex v_{56}, the edge v_1v_2 and the edge v_5v_6 (if it exists). The vertex set of H can be viewed as the union of three paths from v_3 to v_4 (one going through v_6, one through v_1 and the last one through v_2 and v_5). Since H has an even number of vertices, so does one of these paths, and a perfect matching of H (denoted M) can easily be constructed.
- If neither of the two triangles contains a vertex of degree 2 and G contains vertices v_{14}, v_{25} and v_{36}, we choose M to be the matching $\{v_1v_{14}, v_2v_3, v_{25}v_5, v_4v_6\}$.
- If neither of the two triangles contains a vertex of degree 2 and G contains exactly one of the three vertices v_{14}, v_{25} and v_{36} (say, v_{14}), we choose M to be the matching $\{v_1v_3, v_2v_5, v_4v_6\}$.

In all cases, M is clearly a near perfect matching of G that saturates all the main vertices. Let T be a subset of V that induces a 2-connected subgraph of G. Observe that if $|T\cap V^*|\leq 3$, $|M\cap\delta(T\cup N(T))|$ is obviously at most 3, while if $|T\cap V^*|=5$, $T\cap V^*$ is of the form $V^*\setminus\{v_i\}$ for some i and the same conclusion follows. If $|T\cap V^*|=4$, $T\cap V^*$ is $\{v_2, v_3, v_6, v_5\}$, $\{v_1, v_2, v_5, v_4\}$ or $\{v_1, v_3, v_6, v_4\}$. Since M was chosen in such a way that both $M\cap E(\langle v_1v_2\rangle)$ and $M\cap E(\langle v_5v_6\rangle)$ are empty, we conclude that $|M\cap\delta(T\cup N(T))|$ is at most 3 if $|T\cap V^*|=4$. ∎

A matching with the properties of Lemma 5.1 will be called a *standard matching* of G. The previous lemma enables us to prove a theorem similar to Theorem 4.3.

Theorem 5.2. *Let $G=(S,E,w)$ be a triangle-subdivision of the prism, and assume that G is critical, reduced and full. For any standard matching M of G, the fractional chromatic index of $G\setminus M$ (and thus $k(G\setminus M)$) is at most $k(G)-1$.*

Proof. By a familiar argument the maximum valency of $G\setminus M$ is at most $k(G)-1$. Let T be a subset of S inducing a 2-connected critical subgraph of G. Since G is reduced and M is a near perfect matching of G, it suffices to consider those sets T with $5\leq|T|<|S|$. As in the proof of Theorem 4.3, we note that for any two vertices of degree 2, say, v and v', $w(\delta(v))+w(\delta(v'))$ is at least $k(G)$. The only vertex left exposed by M will be denoted by v^*, and the incidence vector of M by y.

Case A. $|N(T)|\leq 1$ and $v^*\notin T$.

Then $|M\cap\delta(T)|=|M\cap\delta(N(T))|+|M\cap\delta(T\cup N(T))|$ is at most $|N(T)|+3$, and hence at most 4. Because all the vertices in T are saturated by M, $|M\cap E(T)|$ is at least $\lfloor |T|/2\rfloor-1$ and $(w-y)(E(T))$ at most $(k(G)-1)\lfloor |T|/2\rfloor$.

Case B. $|N(T)|\geq 2$.

Let v and v' be any two vertices in $N(T)$. Then $(w-y)(\delta(T))$ is at least $k(G)-2$ (by the same argument as in Case G of Theorem 4.5). This implies that $(w-y)(E(T))$ is at most $(k(G)-1)\lfloor |T|/2\rfloor$ by the same argument as in Claim 4.4.

Taken together, Cases A and B prove that

$$(w-y)(E(T)) \leq (k(G)-1)\lfloor |T|/2\rfloor$$

for all sets T not containing v^*.

Case C. $N(T)=\emptyset$ and $v^*\in T$.

Then $|M\cap\delta(T)|$ is at most 3, and thus

$$|M\cap E(T)| \geq \lceil (|T|-1-|M\cap\delta(T)|)/2\rceil \geq \lfloor |T|/2\rfloor -1.$$

We conclude, as in Case A, that $(w-y)(E(T))$ is at most $(k(G)-1)\lfloor |T|/2\rfloor$.

Case D. $|N(T)|=1$ and $v^*\in T$.

Let W denote $S\setminus(T\cup N(T))$, F the subgraph induced by W and L the matching $M\cap E(W)$. Any 2-connected critical subgraph of $F\setminus L$ has an odd set quotient that is at most $k(G)-1$ (because it does not contain v^*). By the remarks of Section 2 the fractional chromatic index of $F\setminus L$ is at most $k(G)-1$. Therefore $w'(E(W))$ is at most $(k(G)-1)\lfloor |W|/2\rfloor$ if we let w' stand for $w-y$. By the same calculation as in Case C of Theorem 4.3, we verify that $2w'(E(T))$ is at most $(k(G)-1)(|T|-1)$.

This completes the proof of the exact conjecture for full graphs that are triangle-subdivisions of the prism. ∎

For triangle-subdivisions of the prism that are nearly full, we need another kind of standard matching.

Lemma 5.3. *Let $G=(V,E,w)$ be a triangle-subdivision of the prism. If the vertex set of G is of the form $S\cup\{v^*\}$, where S induces a 2-connected critical subgraph of G, there exists a matching M in G with the following properties:*

- *M is a perfect matching of G, and*
- *$|M\cap\delta(T\cup N(T))|$ is at most 3 for any subset T of V inducing a 2-connected subgraph of G.*

Proof. Let us consider first the case where at least one of the vertices v_{14}, v_{25} and v_{36} belongs to G. Without loss of generality, we assume that v_{25} belongs to G and let F denote the graph obtained from G by shrinking $\{v_2, v_{25}, v_5\}$. F is a triangle-subdivision of the 4-wheel, and if it has any sector, it must contain an odd number of even sectors. Because G_S is 2-connected, either v^* is v_{25} or v^* does not belong to $\{v_2, v_{25}, v_5\}$. In both cases, if F contained three even sectors, G_S would contain a cycle with 5 (resp. 4, 3) main vertices and 5 (resp. 4, 3) vertices of degree two, contradicting the assumption that G_S is critical. Thus if F has any sector, it has exactly one even sector and we let L denote a standard F-matching of F (as defined in Section 4). If F has no sector, we let L denote any perfect matching of F. We define M as $L \cup \{v_2 v_{25}\}$ if L contains an edge of which v_5 is an endpoint, and $L \cup \{v_{25} v_5\}$ otherwise (see Figure 3 for an illustration).

Clearly $|M \cap \delta(T \cup N(T))|$ is at most 3 if $|T \cap V^*| = 2$, 3 or 5. If $|T \cap V^*| = 4$ and T contains v_2 and v_5, we observe that $|M \cap \delta(T \cup N(T))|$ is at most 3 because M contains edge $v_2 v_{25}$ or $v_{25} v_5$. Otherwise, $T \cap V^*$ is the set $\{v_1, v_3, v_6, v_4\}$, and if F has a sector, its pseudo-vertex is matched to an inner vertex and at least one of v_1, v_3, v_6 and v_4 is not matched to v_2, v_5 or a vertex adjacent to v_2 or v_5. If F has no sector, it contains at most one inner vertex (say, v_{ij}), and vertices in $T \cap V^*$ can be matched only to vertices in $T \cup N(T)$ or $\{v_2, v_5, v_{ij}\}$. In both cases we conclude that $|M \cap \delta(T \cup N(T))|$ is at most 3.

If none of v_{14}, v_{25} and v_{36} belongs to G, the latter contains at most four vertices of degree 2 because S induces a critical subgraph. The proof that the prescribed matching exists is straightforward and we leave it to the reader. ∎

A matching with the properties of Lemma 5.3 will be called a *standard matching* of G.

Theorem 5.4. *Let $G = (V, E, w)$ be a triangle-subdivision of the prism, and assume that G is reduced and nearly full. There exists a matching M in G such that the fractional chromatic index of $G \setminus M$ (and thus $k(G \setminus M)$) is at most $k(G) - 1$.*

Proof. Since G is nearly full, there exists a subset S of G that induces a 2-connected critical subgraph and such that V is $S \cup N(S)$ and $w(E(S))$ is equal to $k(G)\lfloor |S|/2 \rfloor - 1$. If $N(S)$ contains a main vertex, we let v^* denote this vertex. (Note that $\sum_{v \in N(S) \setminus \{v^*\}} w(\delta(v))$ is at most $k(G) - 1$ because $w(\delta(v^*))$ is at least 3.) If $N(S)$ does not contain a main vertex, S must contain all the main vertices and at least one vertex of degree 2; hence $w(\delta(S))$ is at most $k(G)+1$. If $N(S)$ is not empty, we let v^* denote any vertex in $N(S)$ and observe that $\sum_{v\in N(S)\setminus\{v^*\}} w(\delta(v))$ is again at most $k(G)-1$. We choose M to be a standard matching of the subgraph induced by $S\cup\{v^*\}$ (this matching exists by Lemma 5.3). If $N(S)$ is empty, we choose M to be a standard matching of G (see Lemma 5.1). In the rest of the proof we let U denote the set of vertices left unmatched by M. Note that every vertex in U is of degree two, and that if U contains more than one vertex, U is a subset of $N(S)$.

We claim that M is the required matching. First note that since $N(S)$ contains at most one main vertex, the maximum valency of $G\backslash M$ is at most $k(G)-1$. It remains to prove that $(w-y)(E(T)) \leq (k(G)-1)\lfloor |T|/2\rfloor$ for every T inducing a 2-connected critical subgraph of G (where y, as usual, denotes the incidence vector of M). Since G is reduced and the restriction of M to S is a near perfect matching, we may assume that $|T|\geq 5$ and $T\neq S$. The proof is by induction on $|T|$ and $N_s(T)$ denotes the set $\{v\in S\,|\,\delta(v)$ is contained in $\delta(T)\}$.

Case A. $|T\cap U|\leq 1$, $|N_s(T)|\leq 1$ and $w(E(T))\leq k(G)\lfloor |T|/2\rfloor -2$.

By Lemma 5.3 $|M\cap\delta(T\cup N(T))|$ is at most 3, and because $|M\cap\delta(N(T))|$ is at most $|N_s(T)|+1$, the number of vertices of T left unmatched by $M\cap E(T)$ is at most $|T\cap U|+|M\cap\delta(T\cup N(T))|+|N_s(T)|+1\leq 6$. By a familiar argument, $(w-y)(E(T))$ is at most $(k(G)-1)\lfloor |T|/2\rfloor$.

Case B. $|T\cap U|\leq 1$, $|N_s(T)|\leq 1$ and $w(E(T))=k(G)\lfloor |T|/2\rfloor -1$.

Clearly $|T\cap V^*|$ cannot be equal to 3, because if it were, the subgraph induced by T would have two vertices of degree 2 and $T\cup N(T)$ could be shrunk by Corollary 3.5. Similarly, if $|T\cap V^*|$ is equal to 4, T cannot have two vertices of degree 2, and if $|T\cap V^*|=4$ and $|T|=5$, $N(T)$ must be empty. Hence $|T\cap U|+|M\cap\delta(T\cup N(T))|+|N(T)|$ is at most 4 if $|T\cap V^*|=4$. It follows that $(w-y)(E(T))$ is at most $(k(G)-1)\lfloor |T|/2\rfloor$, and the same conclusion holds if $|T\cap V^*|$ is equal to 5 or 6, unless $|T\cap V^*|$ and $|T|$ are equal to 5, $|M\cap\delta(T\cup N(T))|$ is equal to 3 and $|N_s(T)|$ to 1 and v^* belongs to $N(T)\setminus S$ (where v^*, as above, denotes the only matched vertex in $V\setminus S$).

If T has all these properties, let W denote the set of vertices in $S\setminus T$ belonging to edges of $M\cap\delta(T)$. One can show that $|W|=4$ and W contains a main vertex (say, v_1). But then we have $w(\delta(T))\geq w(\delta(S\setminus W,W))+w(\delta(v^*))=w(E(S))-w(E(S\setminus W))-w(E(W))+w(\delta(v^*))\geq \{k(G)\lfloor |S|/2\rfloor -1\}-\{k(G)\lfloor |S\setminus W|/2\rfloor -1\}-\{w(\delta(v_1)-1\}+w(\delta(v^*))\geq k(G)+3$. We conclude that $(w-y)(\delta(T))\geq k(G)-2$, and thus $(w-y)(E(T))\leq (k(G)-1)\lfloor |T|/2\rfloor$ by the same argument as in Claim 4.4.

Case C. $|T\cap U|\geq 2$.

In this case U is a subset of $N(S)$ and is actually the set $N(S)\setminus\{v^*\}$. Let v' and v'' be any two vertices in $N(S)\setminus\{v^*\}$. Since $\sum_{v\in N(S)\setminus\{v^*\}} w(\delta(v))$ is at most $k(G)-1$, we have

$$\begin{aligned} w(E(T)) &\leq w(E(T\setminus\{v',v''\}))+k(G)-1 \\ &\leq (k(G)-1)\lfloor(|T|-2)/2\rfloor+k(G)-1 \\ &= (k(G)-1)\lfloor|T|/2\rfloor \end{aligned}$$

by the induction hypothesis.

Case D. $|N_s(T)|\geq 2$.

Let v and v' be two vertices in $N_s(T)$. Note that $w(\delta(v))+w(\delta(v'))\geq k(G)$ (otherwise the odd set quotient of the subgraph induced by $S\setminus\{v,v'\}$ would be equal to $k(G)$, contradicting the hypothesis that G is reduced). Therefore $(w-y)(\delta(T))\geq k(G)-2$ and $(w-y)(E(T))\leq(k(G)-1)\lfloor|T|/2\rfloor$ by the same argument as in Claim 4.4. ∎

In the following lemma we describe the matching that will be used when G is neither full nor nearly full.

Lemma 5.5. *Let $G=(V,E,w)$ be a triangle-subdivision of the prism. There exists a matching M in G that saturates all vertices of V^* and verifies the following property for every subset S inducing a 2-connected subgraph of G:*

1. $|M\cap E(S)|\geq 1$ *if* $|S\cap V^*|=4$,
2. $|M\cap E(S)|\geq 2$ *if* $|S\cap V^*|=5$, *and*
3. $|M\cap E(S)|\geq 3$ *if* $|S\cap V^*|=6$.

Proof. The lemma can be verified by inspection of the following cases.

- If edges v_1v_4, v_2v_5 and v_3v_6 belong to G, we let M be $\{v_1v_4,v_2v_5,v_3v_6\}$.
- If exactly two edges among v_1v_4, v_2v_5 and v_3v_6 belong to G, say, v_1v_4 and v_2v_5, and if edges v_2v_3 and v_5v_6 also belong to G, we let M be $\{v_1v_4,v_2v_3,v_5v_6\}$.
- If exactly two edges among v_1v_4, v_2v_5 and v_3v_6 belong to G (say, v_1v_4 and v_2v_5), and if one of the edges v_2v_3 and v_5v_6 is missing from G, we let M be $\{v_1v_4,v_2v_5,v_{23}v_3,v_{36}v_6\}$ if v_2v_3 is missing and $\{v_1v_4,v_2v_5,v_3v_{36},v_{56}v_6\}$ otherwise.
- If one and only one of v_1v_4, v_2v_5 and v_3v_6 belongs to G (say, v_1v_4), we let M be $\{v_1v_4,v_2v_3,v_{25}v_5,v_{36}v_6\}$ if v_2v_3 belongs to G, $\{v_1v_4,v_5v_6,v_2v_{25},v_3v_{36}\}$ if v_5v_6 belongs to G and $\{v_1v_4,v_2v_{25},v_5v_{56},v_6v_{36},v_3v_{23}\}$ if neither v_2v_3 nor v_5v_6 belongs to G.

- If none of v_1v_4, v_2v_5 and v_3v_6 belongs to G, and there is at least one vertex of the form v_{ij} for $i,j \in \{1,2,3\}$ or $i,j \in \{4,5,6\}$, we assume (without loss of generality) that v_{12} belongs to G. Then we let M be $\{v_{14}v_4, v_{25}v_5, v_{36}v_6, v_1v_{12}, v_2v_3\}$ if v_2v_3 belongs to G, $\{v_{14}v_4, v_{25}v_5, v_{36}v_6, v_1v_3, v_{12}v_2\}$ if v_1v_3 belongs to G and $\{v_{14}v_4, v_{25}v_5, v_{36}v_6, v_1v_{12}, v_2v_{23}, v_3v_{13}\}$ otherwise.
- If none of v_1v_4, v_2v_5 and v_3v_6 belongs to G, and there is no vertex of the form v_{ij} for $i,j \in \{1,2,3\}$ or $i,j \in \{4,5,6\}$, we choose M to be $\{v_1v_{14}, v_{25}v_5, v_2v_3, v_4v_6\}$. ∎

Theorem 5.6. *Let $G=(V,E,w)$ be a triangle-subdivision of the prism that is reduced but neither full nor nearly full. There exists a matching M in G such that the fractional chromatic index of $G\setminus M$ (and thus $k(G\setminus M)$) is at most $k(G)-1$.*

Proof. Let M be a matching verifying the property of Lemma 5.5. The maximum valency of $G \setminus M$ is at most $k(G)-1$ because M saturates all the main vertices. Let y denote the incidence vector of M and S be any odd set inducing a 2-connected critical subgraph of G. Then $e(S) \doteq w(E(S)) - (k(G)-1)\lfloor |S|/2 \rfloor$ denotes the "excess" of S, i.e., the minimum value that $|M \cap E(S)|$ must take in order for $(w-y)(E(S))$ to be at most $(k(G)-1)\lfloor |S|/2 \rfloor$. The enumeration of the following cases shows that $|M \cap E(S)|$ is at least $e(S)$ for any set S. Observe that if $|S \cap V^*|=5$, $|S|$ is at most 9 (otherwise S would not induce a critical subgraph). Similarly, $|S|$ is at most 11 whenever $|S \cap V^*|=6$.

- $|S\cap V^*|=4$ and $|S|=5$
  Then $e(S)\leq 1$ because the subgraph induced by S is not full.
- $|S\cap V^*|=4$ and $|S|=7$
  $e(S)$ is not equal to 3 because the subgraph induced by S is not full. It is not equal to 2 either, because if it were, $w(E(S))$ would be equal to $k(G)\lfloor |S|/2 \rfloor -1$ and G could be decomposed into smaller multigraphs by Corollary 3.5 (note that S contains three vertices of degree 2). Thus $e(S)$ is at most 1.
- $|S\cap V^*|=5$ and $|S|=5$
  Then $e(S)\leq 1$ because the subgraph induced by S is not full.
- $|S\cap V^*|=5$ and $|S|=7$ or 9
  $e(S)$ is not equal to 4 because the subgraph induced by S is not full. It is not equal to 3 either, because if it were, $w(E(S))$ would be equal to $k(G)\lfloor |S|/2 \rfloor$ or $k(G)\lfloor |S|/2 \rfloor -1$ and S could be shrunk (note that S contains at least two vertices of degree 2). Thus $e(S)$ is at most 2.

- $|S \cap V^*| = 6$ and $|S| = 7$
  Then $e(S) \leq 2$ because the subgraph induced by S is not full.
- $|S \cap V^*| = 6$ and $|S| = 9$ or 11
  If $|S| = 9$, $e(S)$ is less than 4 (otherwise G would be full or S could be shrunk). If $|S| = 11$, $e(S)$ is not equal to 5 for the same reason, and if it were equal to 4, G would be nearly full. Therefore $e(S)$ is at most 3.

We conclude that if M is a matching verifying the property of Lemma 5.5, $|M \cap E(S)|$ is at least $e(S)$ and therefore $(w-y)(E(S))$ is at most $(k(G)-1)\lfloor |S|/2 \rfloor$. ∎

6. Graphs that contain neither $K_5 \setminus e$ nor $K_{3,3}$ as a minor

We can now deal with the general situation, that is, the case where G contains no minor isomorphic to $K_5 \setminus e$ or $K_{3,3}$. Results of Wagner [15] imply that either G has a 1-vertex or 2-vertex cutset, or the graph underlying G is isomorphic to K_n (for $1 \leq n \leq 3$), the prism or the n-wheel for $n \geq 3$. It is not difficult to conclude (see Marcotte [6]) that if G is 2-connected and has at least 6 vertices, then either

- G has a proper 2-vertex cutset (i.e., a cutset whose removal splits the graph into two subsets of vertices V_1 and V_2 with $|V_1|, |V_2| \geq 2$), or
- G is a triangle-subdivision of the triangle, the prism or the n-wheel for $n \geq 3$.

A triangle-subdivision of the triangle is a series-parallel multigraph, and as we noted in the introduction, Seymour [13] has proved that $\chi'(G) = k(G)$ for all such multigraphs. The results of Sections 4 and 5 imply that if G is a triangle-subdivision of the prism or the n-wheel and the exact conjecture holds for all graphs G' such that $V(G') < V(G)$ or $k(G') < k(G)$, then $\chi'(G)$ is also equal to $k(G)$. In this section we assume that G has a proper 2-vertex cutset and briefly recall the technique for decomposing G described in Marcotte [7].

Definition 6.1. Given a 2-connected multigraph $G = (V, E, w)$, we say that $\{u, v\}$ is a *2-vertex cutset* of G if there exist subsets V_L and V_R of V such that

1. $V_L \cap V_R = \{u, v\}$,
2. $E = E(V_L) \cup E(V_R)$,
3. $|V_L| \geq 3$ and $|V_R| \geq 3$.

If the last condition is replaced by the condition "$|V_L| \geq 4$ and $|V_R| \geq 4$", we say that $\{u,v\}$ is a *proper 2-vertex cutset* of G.

Note that if $\{u,v\}$ is a 2-vertex cutset of G, we may assume that u and v are of valency $k(G)$. For if u (say) is not of valency $k(G)$, we may add to G a vertex u' (called a *leaf*) and an edge uu' of multiplicity $k(G)-w(\delta(u))$. This will increase the number of vertices of G, but as we shall see below, this increase will not affect the induction argument. Thus the fractional chromatic index of G is exactly $k(G)$, and as we observed in Section 2, Edmonds' theorem (applied to G) states that w can be expressed as $\sum_{i=1}^{p} \alpha_i x^i$, where $\alpha_i \geq 0$ for all i, $\sum_{i=1}^{p} \alpha_i = k(G)$ and x^i (for $i=1,2,\ldots,p$) is the incidence vector of the matching M_i. Furthermore, because of the assumption that $w(\delta(u))=w(\delta(v))=k(G)$, each of u and v is saturated by M_i for all i.

We wish to decompose G into two multigraphs G_L and G_R such that $k(G_L) \leq k(G)$, $k(G_R) \leq k(G)$ and $\chi'(G) \leq \max(\chi'(G_L), \chi'(G_R))$. Since we want the fractional edge-colourings of G_L and G_R to reflect the given fractional edge-colouring of G, we define the vertex set of G_L (resp. G_R) as $V_L \cup \{v^L\}$ (resp. $V_R \cup \{v^R\}$), where v^L and v^R are pseudo-vertices representing the "rest of the multigraph". The multigraphs G_L and G_R will be called the *left lobe* and *right lobe* of G, respectively.

To determine the multiplicities of the edges incident upon the new vertices, observe that each M_i (for $i=1,2,\ldots,p$) falls into one of the following categories:

1. the matchings M_i that contain the edge uv,
2. the matchings M_i that contain edges ua and vb for $a \in V_L \setminus \{u,v\}$ and $b \in V_L \setminus \{u,v\}$,
3. the matchings M_i that contain edges ua and vb for $a \in V_L \setminus \{u,v\}$ and $b \in V_R \setminus \{u,v\}$,
4. the matchings M_i that contain edges ua and vb for $a \in V_R \setminus \{u,v\}$ and $b \in V_L \setminus \{u,v\}$, and
5. the matchings M_i that contain edges ua and vb for $a \in V_R \setminus \{u,v\}$ and $b \in V_R \setminus \{u,v\}$.

Each of the M_i must be "split" into a matching of G_L and a matching of G_R. If M_i belongs to the first category, it will be split into $M_i \cap E(V_L)$ and $M_i \cap E(V_R)$. If it belongs to the second category, it will be split into $M_i \cap E(V_L)$ and $(M_i \cap E(V_R)) \cup \{uv\}$ (even if uv does not belong to G), and if it belongs to the third category, into $(M_i \cap E(V_L)) \cup \{vv^L\}$ and $(M_i \cap E(V_R)) \cup \{uv^R\}$. Thus to every M_i of the third category correspond a matching of G_L that contains vv^L and a matching of G_R that contains uv^R. It is therefore natural to define G_L (resp. G_R) in such a way that the multiplicity of vv^L (resp.

uv^R) is equal to $\sum(\alpha_i \mid M_i$ belongs to the third category). The difficulty, of course, is that the latter quantity need not be an integer. One can make similar remarks about matchings of the fourth and fifth categories.

We can now complete the definition of G_L and G_R. Let
$\alpha = \sum(\alpha_i \mid M_i$ belongs to the first category),
$\beta = \sum(\alpha_i \mid M_i$ belongs to the second category) and
$\epsilon = \sum(\alpha_i \mid M_i$ belongs to the fifth category).
Observe that α is an integer, because it is the multiplicity of the edge uv. (If uv does not belong to G, we may of course assume that its multiplicity is equal to 0.) G_L is constructed by adjoining the pseudo-vertex v^L to V_L, and by adding $\lceil\epsilon\rceil$ to the multiplicity of uv. Similarly, G_R is constructed by adjoining the pseudo-vertex v^R to V_R, and by adding $\lceil\beta\rceil$ to the multiplicity of uv.

Definition 6.2. G_L is the multigraph $(V_L \cup \{v^L\}, E(V_L) \cup \{uv, uv^L, vv^L\}, w^L)$, where
$w_e^L = w_e$ for $e \notin \{uv, uv^L, vv^L\}$,
$w_e^L = \alpha + \lceil\epsilon\rceil$ if $e = uv$,
$w_e^L = k(G) - \alpha - \lceil\epsilon\rceil - \sum(w_f \mid f = ua$ and $a \in V_L \setminus \{v\})$ if $e = uv^L$, and
$w_e^L = k(G) - \alpha - \lceil\epsilon\rceil - \sum(w_f \mid f = vb$ and $b \in V_L \setminus \{u\})$ if $e = vv^L$.
G_R is the multigraph $(V_R \cup \{v^R\}, E(V_R) \cup \{uv, uv^R, vv^R\}, w^R)$, where
$w_e^R = w_e$ for $e \notin \{uv, uv^R, vv^R\}$,
$w_e^R = \alpha + \lceil\beta\rceil$ if $e = uv$,
$w_e^R = k(G) - \alpha - \lceil\beta\rceil - \sum(w_f \mid f = ua$ and $a \in V_R \setminus \{v\})$ if $e = uv^R$, and
$w_e^R = k(G) - \alpha - \lceil\beta\rceil - \sum(w_f \mid f = vb$ and $b \in V_R \setminus \{u\})$ if $e = vv^R$.

The above definition "works", in spite of the rounding. The following theorems may be found in Marcotte [7], where they appear as Theorems 2.10 and 2.11, respectively. Observe that by construction of G_L and G_R, the valency of u (resp. v) in each of these multigraphs is equal to $k(G)$. Also the valency of any vertex of G_L (resp. G_R) is at most $k(G)$. Finally, the choice of the multiplicities for uv, uv^L, vv^L, uv^R and vv^R implies that the maximum odd set quotient of G_L (resp. G_R) is at most $k(G)$. This follows because the edge multiplicities have been defined by using an optimal fractional colouring of G.

Theorem 6.3. *The fractional chromatic indices of G_L and G_R are equal to $k(G)$.*

Theorem 6.4. *Let G_L and G_R be defined as in Definition 6.2, and let $C_1 = \{N_1, N_2, \ldots, N_{k(G)}\}$ be an edge-colouring of G_L and $C_2 = \{Q_1, Q_2, \ldots, Q_{k(G)}\}$ an edge-colouring of G_R, where the N_i and Q_j are matchings in G_L and G_R*

respectively. Then there exists an edge-colouring $C=\{M_1,M_2,\ldots,M_{k(G)}\}$ *of* G *such that for every* $t\in\{1,2,\ldots,k(G)\}$, $M_t\cap E(V_L)$ *is contained in* $N_i\cap E(V_L)$ *for some* i *and* $M_t\cap E(V_R)$ *in* $Q_j\cap E(V_R)$ *for some* j.

These theorems enable us to conclude, by induction on $|V(G)|$, that all multigraphs with no minor isomorphic to $K_5\setminus e$ or $K_{3,3}$ verify the exact conjecture.

Theorem 6.5. *The chromatic index of* G *is* $\max\{\rho,\lceil\kappa\rceil\}$ *for any graph* G *that does not contain* $K_5\setminus e$ *or* $K_{3,3}$ *as a minor.*

Proof. The proof is by induction on $|V(G)|$ and $k(G)$. Note that the theorem is true for graphs with 10 vertices or fewer (see Plantholt and Tipnis [8]). If G has a cut vertex, the edge-colourings of the 2-connected components of G can be combined to yield an edge-colouring of G itself. If G is 2-connected but not reduced, the results of Section 3 and the induction hypothesis imply that $\chi'(G)=\max\{\rho,\lceil\kappa\rceil\}$ (note that the multigraphs obtained from G by shrinking connected subgraphs of G cannot contain $K_5\setminus e$ or $K_{3,3}$ as a minor). If G is a triangle-subdivision of the triangle, the result of Seymour [13] (for series-parallel multigraphs) implies that $\chi'(G)=\max\{\rho,\lceil\kappa\rceil\}$. If G is reduced and is a triangle-subdivision of the prism or the n-wheel for some $n\geq 3$, the results of Sections 4 and 5 imply that there exists a matching M such that $k(G\setminus M)=k(G)-1$. The induction hypothesis then yields $\chi'(G)=\chi'(G\setminus M)+1=k(G)$.

Finally, if G is 2-connected and reduced and has a proper 2-vertex cutset, we let $\{u,v\}$ denote such a cutset and V_L and V_R the corresponding subsets of vertices. The following procedure will yield an edge-colouring of G with the desired properties:

- add leaves u' and v' to G (if necessary) so that $w(\delta(u))=w(\delta(v))=k(G)$,
- decompose G into lobes G_L and G_R, where the vertex set of G_L is $V_L\cup\{v^L\}$ and the vertex set of G_R consists of the vertices in $V_R\cup\{v^R\}$ and the leaves (if any),
- colour the edges of $G_L\setminus\{u',v'\}$ and $G_R\setminus\{u',v'\}$ with at most $k(G)$ colours,
- extend the edge-colouring of $G_L\setminus\{u',v'\}$ (resp. $G_R\setminus\{u',v'\}$) to an edge-colouring of G_L (resp. G_R), and
- "fuse" the colourings of G_L and G_R to obtain a colouring of G (see Theorem 6.4).

Note that the induction hypothesis can be applied to $G_L\setminus\{u',v'\}$ and $G_R\setminus\{u',v'\}$ because they have fewer vertices than G, and that the edge-colouring of $G_L\setminus\{u',v'\}$ (resp. $G_R\setminus\{u',v'\}$) can be extended to an edge-colouring of G_L (resp. G_R) because u' and v' are leaves. ∎

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